

Semiclassical Analysis of the Supershell Effect in Reflection-Asymmetric Superdeformed Oscillator

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Abstract

An oscillatory pattern in the smoothed quantum spectrum, which is unique for single-particle motions in a reflection-asymmetric superdeformed oscillator potential, is investigated by means of the semiclassical theory of shell structure. Clear correspondence between the oscillating components of the smoothed level density and the classical periodic orbits is found. It is shown that an interference effect between two families of the short periodic orbits, called supershell effect, develops with increasing reflection-asymmetric deformations. Possible origins of this enhancement phenomena as well as quantum signatures of period-multiplying bifurcations are discussed in connection with stabilities of the classical periodic orbits.

§1. INTRODUCTION

Possible occurrence of instability of superdeformed (SD) nuclei having the prolate shape with the axis ratio approximately 2:1 against the octupole-type reflection asymmetric deformation is one of the current topics of growing interest in high-spin nuclear structure physics. Regions in the (N, Z) plane where we can expect existence of reflection-asymmetric SD nuclei have been investigated^{[1]~[5]} mainly by means of the Strutinsky-type calculations of the collective potential energy surface (see also Refs. [6] and [7] for other approaches). Concerning the physical condition for the occurrence of the octupole instability, Nazarewicz and Dobaczewski^[8] have recently discussed the connection between the dynamical symmetry of the anisotropic harmonic-oscillator with frequencies in rational ratio and the multi-cluster configurations. They have suggested

that the closed-shell configurations in the prolate SD oscillator potential, defined as having the frequency ratio $\omega_{\perp}/\omega_z = 2$, might be unstable (stable) against the octupole-type reflection asymmetric shapes when the single-particle levels are filled up to the major shells with N_{sh} =even (odd), N_{sh} being the shell quantum number defined by $N_{\text{sh}} = 2n_{\perp} + n_z$ (see also the previous work, Ref. [9]). Their suggestion is in good qualitative agreement with the realistic shell-structure energy calculation by Höller and Åberg.^[2] We call the N_{sh} -dependence of the octupole instability “odd-even effect in N_{sh} ”.

We have suggested in Refs. [10], [11] a possible relationship between the odd-even effect mentioned above and the “supershell effect” in reflection-asymmetric SD potentials. The general concept of supershell was originally introduced by Balian and Bloch^[12] in relation to the semi-classical theory of shell structure. Quite recently, the supershell effect has been observed, for the first time, in the mass abundance spectra of metal clusters. Theoretical analysis of this phenomena has been made by Nishioka, Hansen and Mottelson.^{[13],[14]}

As is well known, clustering of eigenvalues, that is, oscillating pattern in the energy-smoothed level density for single-particle motions in the mean field is called shell structure.^[15] In the semi-classical theory, classical periodic orbits having relatively short periods are responsible for the clustering of the levels; the frequencies in the level density oscillation is determined by the corresponding periods of classical motion.^{[16]~[23]} When two families of short period orbits interfere and produce an undulating pattern in the oscillating level density, this pattern is called supershell structure.^{[12],[16]} In the case of the metal clusters, a beautiful beating pattern enveloping individual shell oscillations which is caused by the interference between the triangular and square orbits of an electron in a spherical Woods-Saxon potential has been demonstrated^{[13],[14]} to nicely correspond to the experimental data.

In the case of the SD nuclei under consideration, an interference effect is expected to arise between the classical periodic orbits with period $T \approx 2\pi/\omega_{\perp}$ and those with $T \approx 2\pi/\omega_z$ of a nucleon in the reflection-asymmetric SD potential.^[11] The main purpose of this paper is to show that the interference effect brings about another example of the supershell structure, which is intimately connected with the odd-even effect in N_{sh} mentioned above, and which is relevant to experimental investigations. It should be emphasized here that, contrary to the case of spherical potentials, the Hamiltonian describing the single-particle motions in a deformed mean field is non-integrable in general. Accordingly, our Hamiltonian system is a kind of mixed system whose phase space is composed both of regular and chaotic regions. As a consequence, properties of our

phase space change in a quite sensitive manner when the shape of the nuclear surface is varied. In this paper, we shall show that the supershell structure becomes more significant with increasing octupole deformation. Possible origins of this enhancement will be discussed in relation to the change in the properties of the classical periodic orbits as a function of the octupole deformation parameter.

After briefly reviewing the semiclassical theory of shell structure in §2, we first apply in §3 both the torus quantization method and the periodic-orbit quantization method to the case of the prolate SD oscillator potential. In this integrable limit, the supershell effect can be treated analytically. In §4, a reflection-asymmetric SD potential model is introduced and the supershell pattern in the quantum level spectrum calculated for this potential is exhibited. In §5, we investigate properties of classical motions in this potential, like stabilities and bifurcation phenomena of the periodic orbits. In §6, we show that a nice correspondence holds between peak positions of the Fourier transform of quantum spectrum and periods of classical closed orbits; relative heights between peaks change as functions of the octupole-deformation parameter, providing us with a semiclassical interpretation of the origin and the octupole-deformation dependence of the supershell structure. Here, quantum signature of the bifurcations is also discussed. Summary of this work is given in §7.

A preliminary version of this work was previously reported in this journal.^[24]

§2. SOME ELEMENTS OF SEMICLASSICAL THEORY OF SHELL STRUCTURE

In this section we briefly review some basic elements of the semiclassical theory of shell structure, which are necessary for later discussions.

2.1. Torus Quantization

To begin with, let us consider the case of multi-dimensional, integrable Hamiltonian system, where the Hamiltonian can be written as a function of only action variables I_i , being independent of angle variables θ_i conjugate to them. Semiclassical quantization condition valid for such systems has been formulated by Einstein-Brillouin-Keller, and called torus quantization or the EBK quantization;

$$I_i(E_{n_1, \dots, n_f}) = \oint_i \mathbf{p} d\mathbf{q} = \hbar(n_i + \alpha_i/4) , \quad i = 1, \dots, f \quad (2.1)$$

where indices i represent mutually independent paths on f -dimensional torus constructed by classical trajectories, α_i are Maslov indices related to the singularities of the Van Vleck determinant appearing in the semiclassical propagator along the path i . Thus, the semiclassical level density is given by

$$g(E) = \sum_{\{n\}} \delta(E - H(I_i = \hbar(n_i + \alpha_i/4))) . \quad (2.2)$$

The summation on the r.h.s. may be rewritten using the Poisson sum formula into the form of topological sum over periodic orbits.^[25] In Ref. [25], spherical systems are analyzed and clear correspondence between the topological sum and the periodic orbits is shown. In the spherical case, all periodic orbits satisfy the resonance condition, *i.e.*, the frequency ratio of radial and angular motions are the same as that of topological indices. We shall apply this method to the SD harmonic oscillator potential in §3, and discuss the correspondence between the topological indices and periodic orbits. There, it will be shown that some “partially-resonant” terms play an important role giving rise to the supershell effect.

2.2. Periodic-Orbit Quantization

Next, let us consider the case of multi-dimensional non-integrable Hamiltonian system. For such systems, as is well known, the periodic-orbit quantization method provides us with a useful base toward understanding the correspondence between classical periodic orbits and properties of quantum spectra.^{[22], [26]} This theory is essentially based on the path integral formalism of quantum mechanics. The first step is to express the quantum level density $g(E) = \sum_n \delta(E - E_n)$ as a trace of the energy-dependent Green function;

$$\begin{aligned} g(E) &= -\frac{1}{\pi} \text{Im Tr} \frac{1}{E + i\epsilon - \hat{H}} \\ &= -\frac{1}{\pi} \text{Im} \int d\mathbf{q} G(\mathbf{q}, \mathbf{q}; E) . \end{aligned} \quad (2.3)$$

The Green function $G(\mathbf{q}'', \mathbf{q}'; E)$ is a Fourier transform of the transition amplitude $K(\mathbf{q}'', t; \mathbf{q}', 0) = \langle \mathbf{q}'' | \exp(-it\hat{H}/\hbar) | \mathbf{q}' \rangle \cdot \theta(t)$, and we can express it in the path integral form. Evaluation of the path integral by the stationary phase approximation (SPA) extracts the classical trajectories.

The Fourier transformation is also performed by means of the SPA. Finally, the trace integral appearing in Eq. (2.3) extracts the periodic orbits and one obtains the following expression called the Gutzwiller trace formula:

$$g(E) \simeq \bar{g}(E) + \sum_{n,\gamma} A_{n\gamma}(E) \cos(nS_\gamma(E)/\hbar - (\pi/2)\mu_{n\gamma}) , \quad (2.4)$$

where $\bar{g}(E)$ denotes the average level density and the second term on the r.h.s. represents the oscillating part. The summation is taken over all periodic orbits and their multiple traversals. S_γ is a classical action along the orbit γ , $S_\gamma = \oint_\gamma \mathbf{p} \cdot d\mathbf{q}$, and μ_γ is a Maslov phase. The amplitude factor $A_{n\gamma}$ depends on the phase space structure about the periodic orbit γ , as we shall discuss in §2.4. For sufficiently isolated orbits, the trace integral is well approximated by the SPA and the amplitude factor for the n -fold traversal of orbit γ can be written as^[26]

$$A_{n\gamma} = \frac{1}{\pi\hbar} \frac{T_\gamma}{\sqrt{|\det(\mathbf{1} - M_\gamma^n)|}} , \quad (2.5)$$

where T_γ and M_γ represent the period and the monodromy matrix of the primitive orbit γ , respectively. This expression is known to work well for chaotic systems such as billiards.^[21]

2.3. Stability of Classical Trajectories

The amplitude factor in the trace formula is related with the properties of phase space around the periodic orbits. Let us write the Hamilton equation in $2f$ -dimensional phase space as

$$\frac{d}{dt}Z = \Lambda \nabla H , \quad (2.6)$$

with

$$Z = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}, \quad \nabla = \begin{pmatrix} \nabla_p \\ \nabla_q \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

and consider the time evolution of the deviation $\delta Z(t)$ from the reference classical trajectory $Z_0(t)$. To the first order in δZ , we obtain

$$\frac{d}{dt}\delta Z = \Lambda \mathcal{H} \delta Z , \quad (2.7)$$

where \mathcal{H} is the Hessian matrix defined by

$$\mathcal{H} = \begin{pmatrix} H_{pp} & H_{pq} \\ H_{qp} & H_{qq} \end{pmatrix}_{Z_0}, \quad (H_{pq})_{ij} = \frac{\partial^2 H}{\partial p_i \partial q_j} \quad \text{etc.} \quad (2.8)$$

Knowing $\mathcal{H}(t)$, Eq. (2.7) can be easily integrated;

$$\delta Z(t) = \exp \left[\Lambda \int_0^t dt' \mathcal{H}(Z_0(t')) \right] \delta Z(0) \equiv \mathcal{M}(t) \delta Z(0). \quad (2.9)$$

\mathcal{M} is called the stability matrix. It is real and symplectic; $\Lambda \mathcal{M}^T \Lambda^{-1} = \mathcal{M}^{-1}$. When one take a periodic orbit as the reference trajectory and the period T as time t , the stability matrix is particularly called monodromy matrix $M_\gamma \equiv \mathcal{M}(T_\gamma)$.^[27] It is known that the eigenvalues of M_γ are independent of the initial condition $Z(0)$ on the orbit. According to the symplectic property of the monodromy matrix, its eigenvalues appear in pairs $(+/-)(e^\alpha, e^{-\alpha})$, where α is real or pure imaginary. When α is pure imaginary ($\alpha = iw$), the orbit is stable and torus exists surrounding it. w is called winding number of the torus. When α is real ($\alpha = \lambda T$), the orbit is unstable and λ is called the Lyapunov exponent which measures the degree of instability.

2.4. Magnitudes of Shell Effects

Strength of the shell effect depends mainly on three factors, to be discussed below, associated with the periodic orbits.^[18]

The first factor is the degeneracy of the orbit. Here the term ‘degeneracy’ means the number of independent continuous parameters (additional to energy) that specify a certain orbit from continuous family of orbits having the same action. For example, planar orbits in a spherical potential form continuous family generated by rotation and the degeneracy is generally 3, since a certain orbit belonging to this family is specified by three Euler angles. As illustrated by the above example, degeneracy is related to continuous symmetry of the system. These degeneracies correspond to the unit eigenvalues of M .

The second factor is the stability of the orbit. For non-integrable systems, evaluating the trace integral by the SPA, one sees that the amplitude behaves as

$$A_{n\gamma} \propto \frac{1}{\sqrt{|\det(\mathbf{1} - \tilde{M}_\gamma^n)|}}, \quad (2.10)$$

where \tilde{M} is a reduced monodromy matrix in which degrees of freedom corresponding to the unit eigenvalues of M are excluded out. The more unstable is the orbit, the weaker is its contribution

to the level density, because it has a large Lyapunov exponent and the denominator on the r.h.s. becomes large. The above proportionality is valid only when all the eigenvalues of \tilde{M} are sufficiently distant from unity. However, one of the eigenvalues may happen to be very close to unity. This is called nonlinear resonance where two frequencies of the torus coincide with each other and gives rise to a periodic orbit bifurcation. Namely, the period n -upling bifurcation occurs when $\det(\mathbf{1} - \tilde{M}_\gamma^n) = 1$. In this resonance region, one have to use a more sophisticated treatment than the SPA; for example, the uniform approximation using the resonant normal form. Such a procedure is formally discussed by Ozorio de Almeida and Hannay,^[28] but, to the best of our knowledge, application of this theory to multi-dimensional, non-integrable Hamiltonian system has not been performed yet.

The third factor is the phase space volume occupied by an orbit. It is not important in our analysis because it is insensitive to the variation of potential parameters.

Let us examine how these three factors enter in the amplitude factors for different types of periodic orbits. First, consider a chaotic orbit, that is, a well-isolated orbit whose degeneracy equals zero. Its amplitude factor is given by Eq. (2.5). There appears the same stability factor as Eq. (2.10), and the period T measures the phase space volume of the orbit. Next, as an example of non-isolated orbits, let us consider orbits in axially symmetric deformed potentials, whose degeneracy equals one corresponding to the rotation about the symmetry axis. It is convenient to use the cylindrical coordinates (ρ, φ, z) . One should then perform the integral uniformly with respect to φ in the trace, because in this direction periodic orbits exist continuously. Thus we obtain the following expression of the amplitude factor for these orbits (see Appendix A):

$$A_{n\gamma}(E) = \frac{4\pi}{(2\pi\hbar)^{3/2}} \frac{B_\gamma}{\sqrt{|2 - \text{Tr}\tilde{M}_\gamma^n|}}, \quad (2.11)$$

$$B_\gamma = \int_0^{T_\gamma} dt \left| \frac{\partial\varphi(t + T_\gamma)}{\partial p_\varphi(t)} \right|^{-1/2}.$$

Here, B_γ contains the first and the third factors mentioned above. The period n -upling bifurcation occurs when $\text{Tr}\tilde{M}_\gamma^n = 2$.

§3. SUPERSHELL EFFECT IN THE SD OSCILLATOR

In this section we apply the semiclassical theories to the axially-symmetric 2:1 deformed harmonic oscillator Hamiltonian

$$H_0(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p}^2}{2M} + \sum_{i=x,y,z} \frac{M\omega_i^2 q_i^2}{2}, \quad (3.1)$$

where $\omega_x = \omega_y \equiv \omega_\perp = 2\omega_z \equiv 2\omega_{\text{sh}}$, and we discuss how the supershell structure emerges in this case. We compare the two semiclassical quantization methods summarized in the preceding section, and discuss their relations.

3.1. The Periodic Orbit Method

In this subsection we analyze the supershell effect in the SD oscillator defined by (3.1) using the Gutzwiller trace formula. The trace formula can be derived also for such an integrable system if the degeneracy (mentioned below) is properly taken into account. According to Ref. [29], the semiclassical level density may be written as

$$g(E) \simeq \bar{g}(E) + g_{\text{osc}}^{(\text{IV})}(E) + g_{\text{osc}}^{(\text{II})}(E). \quad (3.2)$$

The first term on the r.h.s. represents the mean level density;

$$\bar{g}(E) = \frac{1}{(2\pi\hbar)^3} \int d\mathbf{p} d\mathbf{q} \delta(E - H(\mathbf{p}, \mathbf{q})) = \frac{E^2}{8(\hbar\omega_{\text{sh}})^3}. \quad (3.3)$$

The second and the third terms are the oscillating parts representing the shell effects. The superscript (II) and (IV) denote the degeneracies of the periodic orbits. $g_{\text{osc}}^{(\text{IV})}(E)$ is a contribution from four-fold degenerate orbits whose periods are multiples of $T^{(\text{IV})} = 2\pi/\omega_{\text{sh}}$. The present model is very special in the sense that all trajectories are periodic, and one should explicitly perform four integrals (corresponding to the degeneracy) in evaluating the trace formula. Thus one obtains the following expression;

$$\begin{aligned} g_{\text{osc}}^{(\text{IV})}(E) &= \sum_{m \neq 0} \frac{E^2}{8(\hbar\omega_{\text{sh}})^3} \cos \left[m \left(\frac{S^{(\text{IV})}}{\hbar} - (4 + 4 + 2) \frac{\pi}{2} \right) \right] \\ &= \sum_{m=-\infty}^{\infty} \frac{E^2}{8(\hbar\omega_{\text{sh}})^3} \cos \left[m \left(\frac{ET^{(\text{IV})}}{\hbar} - 5\pi \right) \right] - \bar{g}(E) \end{aligned} \quad (3.4)$$

$$= \sum_N \frac{(N + 5/2)^2}{8} \delta \left(E - \hbar\omega_{\text{sh}} \left(N + \frac{5}{2} \right) \right) - \bar{g}(E), \quad (3.5)$$

where $S^{(\text{IV})} = ET^{(\text{IV})}$ is the action integral along the primitive periodic orbit and the sum over m accounts for multiple traversals. The last expression is obtained by using the Poisson sum formula

$$\sum_{m=-\infty}^{\infty} \exp(2\pi i m A) = \sum_{N=-\infty}^{\infty} \delta(A - N) . \quad (3.6)$$

In Eq. (3.2), $g_{\text{osc}}^{(\text{II})}(E)$ is the contribution from two-fold degenerate orbits whose periods are odd integer times $T^{(\text{II})} = T^{(\text{IV})}/2$. It is obtained in a similar manner, except that the integrations with respect to the z direction may be performed by the SPA. The result is written as

$$\begin{aligned} g_{\text{osc}}^{(\text{II})}(E) &= \sum_{m'=\text{odd}} \frac{E}{8(\hbar\omega_{\text{sh}})^2 \sin(m'\pi/2)} \sin \left[m' \left(\frac{S^{(\text{II})}}{\hbar} - (2+2)\frac{\pi}{2} \right) \right] \\ &= \sum_{m=-\infty}^{\infty} \frac{E}{8(\hbar\omega_{\text{sh}})^2} \cos \left[(2m+1) \left(\frac{ET^{(\text{II})}}{\hbar} - \frac{5\pi}{2} \right) \right] \end{aligned} \quad (3.7)$$

$$= \sum_N (-)^N \frac{N+5/2}{8} \delta \left(E - \hbar\omega_{\text{sh}} \left(N + \frac{5}{2} \right) \right) , \quad (3.8)$$

when the sum over m' accounts for multiple traversals of the primitive periodic orbit. The expressions (3.4) and (3.7) were first derived in Ref. [29].

Summing up the above three contributions, we obtain the degeneracy d_N of the N -th shell as

$$d_N = \frac{([N/2] + 1)([N/2] + 2)}{2} + \frac{3}{32} , \quad (3.9)$$

where $[*]$ is the Gauss symbol. The first term on the r.h.s. correspond to the exact degeneracy of the quantum spectrum. We thus see that the result obtained by the trace formula is very accurate (the deviation from the exact quantum result is only $3/32$).

Now let us focus our attention on a smoothed density of levels with finite energy resolution $\delta E = \hbar\omega_{\text{sh}}$. It is then sufficient to consider finite number of periodic orbits of short periods satisfying the following uncertainty relation;

$$T \leq T_{\text{max}} = \frac{2\pi\hbar}{\delta E} . \quad (3.10)$$

As far as gross properties of the level density is concerned, therefore, the well-known problem of the long time propagation in the semiclassical approximation does not occur.

We show in Figs. 1-a) and 1-b) the contributions from the families of periodic orbits with periods $2\pi/\omega_{\text{sh}}$ and $2\pi/\omega_{\perp}$, respectively, and in Fig. 1-c) the sum of them. There appears an undulating pattern in the level density due to the interference of the above two families of periodic orbits, which is just the supershell structure. Thus, one sees that the supershell structure emerges from this interference effect.

3.2. The EBK Method

Defining the action-angle variables (I_i, θ_i) by

$$\begin{aligned} p_i &= \sqrt{2M\omega_i I_i} \sin \theta_i , \\ q_i &= \sqrt{2I_i/M\omega_i} \cos \theta_i , \end{aligned} \quad (3.11)$$

the Hamiltonian (3.1) is written as a function of only action variables as $H_0(\mathbf{I}) = \boldsymbol{\omega} \cdot \mathbf{I}$. The Maslov indices are 2 for all paths i , and therefore the EBK quantization condition becomes

$$I_i = \hbar(n_i + 1/2) . \quad (3.12)$$

In the present case, this EBK quantization gives exact quantum eigenvalues: $E = \sum_i \hbar\omega_i(n_i + 1/2)$. Now let us investigate the roles of classical periodic orbits in giving rise to the supershell structure in the quantum spectrum. For this purpose, we use the method of topological sum.^[25] The semiclassical level density is written as

$$g(E) = \sum_{\{n\}} \delta\left(E - \sum_i \hbar\omega_i(n_i + \alpha_i/4)\right) . \quad (3.13)$$

Using the Poisson sum formula (3.6), one can rewrite Eq. (3.13) as

$$g(E) = g_0(E) + \sum_{M \neq 0} g_M(E) , \quad (3.14)$$

where

$$g_M(E) = \frac{1}{\hbar^3} \int_0^\infty d\mathbf{I} \delta(E - \boldsymbol{\omega} \cdot \mathbf{I}) \exp(2\pi i \mathbf{M} \cdot (\mathbf{I}/\hbar - \boldsymbol{\alpha}/4)) , \quad (3.15)$$

and the summation is taken over all the combinations of integers, $\mathbf{M} = (M_x, M_y, M_z)$. Here, $g_0(E)$ represents a mean level density corresponding to the Thomas-Fermi approximation;

$$g_0(E) = \frac{1}{\hbar^3} \int_0^\infty d\mathbf{I} \delta(E - \boldsymbol{\omega} \cdot \mathbf{I}) = \frac{E^2}{8(\hbar\omega_{\text{sh}})^3} . \quad (3.16)$$

The remaining terms with non-zero \mathbf{M} represent the oscillating part responsible for the shell structure. To simplify the expression, we introduce the notation $f_i = 2\pi M_i \omega_{\text{sh}}/\omega_i$. The dominant contribution comes from terms satisfying the resonance condition $\mathbf{M} = \mathbf{M}^* \propto \boldsymbol{\omega}$, *i.e.*, $f_x = f_y =$

f_z (in the present case, $\mathbf{M} = m(2, 2, 1)$). Carrying out the integration with respect to \mathbf{I} and denoting the sum over such resonant terms as $g_{\text{osc}}^{(\text{i})}$, we obtain

$$g_{\text{osc}}^{(\text{i})}(E) = \frac{E^2}{8(\hbar\omega_{\text{sh}})^3} \sum_{m \neq 0} e^{2\pi i m(E/\hbar\omega_{\text{sh}} - 5/2)}. \quad (3.17)$$

Next, let us consider the ‘partially resonant’ terms which satisfy the condition $f_i = f_j \equiv f_{\perp} \neq f_k$. Carrying out the integration with respect to \mathbf{I} , they are evaluated as

$$\begin{aligned} g_{M(\text{part.res.})}(E) &= \frac{1}{4(\hbar\omega_{\text{sh}})^2} \left(\frac{E}{i(f_{\perp} - f_k)} e^{if_{\perp}E/\hbar\omega_{\text{sh}}} + \frac{\hbar\omega_{\text{sh}}}{(f_{\perp} - f_k)^2} (e^{if_{\perp}E/\hbar\omega_{\text{sh}}} - e^{if_kE/\hbar\omega_{\text{sh}}}) \right) e^{i\pi M \cdot \alpha/2} \\ &\simeq \frac{1}{4(\hbar\omega_{\text{sh}})^2} \frac{E}{i(f_{\perp} - f_k)} e^{if_{\perp}E/\hbar\omega_{\text{sh}}} e^{i\pi M \cdot \alpha/2}, \end{aligned} \quad (3.18)$$

where the second term on the r.h.s. is neglected because it is higher-order in \hbar . Let us then take terms with $f_x = f_y \equiv f_{\perp} \neq f_z$, and write \mathbf{M} as (m', m', l) . Summing over terms with odd- m' and arbitrary integer l , we obtain

$$\begin{aligned} g_{\text{osc}}^{(\text{ii})}(E) &= \sum_{m'=\text{odd}} \sum_{l=-\infty}^{\infty} \frac{1}{4(\hbar\omega_{\text{sh}})^2} \frac{E}{2\pi i(m'/2 - l)} e^{2\pi i m' E/2\hbar\omega_{\text{sh}}} \cdot e^{-2\pi i \{2m' + l\}/2} \\ &= \frac{E}{8(\hbar\omega_{\text{sh}})^2} \sum_{m=-\infty}^{\infty} e^{i\pi(2m+1)(E/\hbar\omega_{\text{sh}} - 5/2)}. \end{aligned} \quad (3.19)$$

The last expression is obtained using the expansion formula of cosecond in partial fractions:

$$\text{cosec } z = \sum_{l=-\infty}^{\infty} \frac{(-)^l}{z - l\pi}. \quad (3.20)$$

It can be easily shown in a similar way that the sum of other terms which are the same order in \hbar as (3.19) vanishes.

3.3. Relation between the Two Methods

Now, let us discuss the correspondence between the results obtained by the periodic-orbit method and the EBK method. It is evident that the two terms $g_{\text{osc}}^{(\text{i})}(E)$ and $g_{\text{osc}}^{(\text{ii})}(E)$ in the EBK treatment are identical with the contributions $g_{\text{osc}}^{(\text{IV})}(E)$ and $g_{\text{osc}}^{(\text{II})}(E)$ which are evaluated by the trace formula for the periodic orbits with periods $2\pi/\omega_{\text{sh}}$ and $2\pi/\omega_{\perp}$, respectively. This

result is very instructive to understand the physical meanings of the resummation with respect to the indices \mathbf{M} by the use of the Poisson sum formula. The indices satisfying the resonance condition, $\mathbf{M} = m(2, 2, 1)$, correspond to a family of classical orbits with periods $2m\pi/\omega_{\text{sh}}$. One can examine this by comparing Eqs. (3.4) and (3.17). On the other hand, for partially-resonant contributions, we find the correspondence in the following way. Comparing Eqs. (3.7) and (3.19), we notice that the family of planar orbits in the (x, y) plane corresponds to the summation over indices $\mathbf{M} = (1, 1, l)$ with $-\infty < l < \infty$. Its m -fold traversals are related with $\mathbf{M} = (m, m, l)$. These partially-resonant terms play important roles in formation of the supershell structure in the present model.

§4. REFLECTION-ASYMMETRIC SD OSCILLATOR MODEL

4.1. Model Hamiltonian

Let us consider a model Hamiltonian consisting of an axially-symmetric 2:1 deformed harmonic oscillator and a doubly-stretched octupole (Y_{30}) deformed potential;

$$H = \frac{\mathbf{p}^2}{2M} + M\omega_0^2 \left(\frac{r^2}{2} - \lambda_{30} r^2 Y_{30}(\theta) \right)'' , \quad (4.1)$$

where double primes indicate that the variables in parenthesis are defined in terms of the doubly-stretched coordinates $q_i'' = (\omega_i/\omega_0)q_i$, and $\omega_0 = (\omega_x\omega_y\omega_z)^{1/3}$. As emphasized by Sakamoto and Kishimoto^[30], the doubly-stretched coordinates are suited to description of systems having quadrupole equilibrium deformations, and possess several advantages over the usual coordinates; for example, the center of mass motion is exactly decoupled from the octupole-type deformations described by the above Hamiltonian. Note that the doubly-stretched octupole operator is in fact a linear combination of the ordinary dipole and octupole operators, although we sometimes omit the adjective “doubly-stretched” for simplicity. In (4.1), we adopt the quadratic radial dependence for the octupole-deformed potential for the ease of taking into account the volume conservation condition. By requiring the volume surrounded by an equipotential surface to be independent of the octupole deformation parameter λ_{30} , the λ_{30} -dependence of ω_0 is determined as

$$\omega_0(\lambda_{30}) = \omega_0(0) \left[\frac{1}{4\pi} \int d\Omega (1 - 2\lambda_{30} Y_{30}(\Omega))^{-3/2} \right]^{1/3} . \quad (4.2)$$

We note that the average level density $\bar{g}(E)$ is independent of λ_{30} when ω_0 satisfies Eq. (4.2). Let us define dimensionless variables as

$$\begin{cases} p_i \rightarrow \sqrt{M\hbar\omega_0} p_i , \\ q_i \rightarrow \sqrt{\hbar/M\omega_0} q_i , \\ H \rightarrow \hbar\omega_0 H . \end{cases} \quad (4.3)$$

Then the Hamiltonian (4.1) becomes

$$H = \frac{\mathbf{p}^2}{2} + \left(\frac{r^2}{2} - \lambda_{30} r^2 Y_{30}(\theta) \right)'' . \quad (4.4)$$

Since this potential is a homogeneous function of the second order in coordinates, the scaling relation

$$H(\alpha\mathbf{p}, \alpha\mathbf{q}) = \alpha^2 H(\mathbf{p}, \mathbf{q}) \quad (4.5)$$

holds. Thus, if $(\mathbf{p}(t), \mathbf{q}(t))$ is a solution of the Hamilton equation with energy E , $(\alpha\mathbf{p}(t), \alpha\mathbf{q}(t))$ is also a solution but with energy $\alpha^2 E$. Namely, once the classical properties of the system are known on a certain energy surface E_0 , properties on other energy surface E are obtained by scale transforming the phase space variables (\mathbf{p}, \mathbf{q}) to $(\alpha\mathbf{p}, \alpha\mathbf{q})$ with $\alpha = \sqrt{E/E_0}$.

4.2. Supershell Structure

Figure 2 shows the oscillating part of the level density for the Hamiltonian (4.4) calculated by means of the Strutinsky method. A characteristic property of the oscillating level density is that it exhibit the supershell pattern. Figure 3 gives a phenomenological illustration of the concept of the supershell. It is seen from this figure that the oscillating level density can be represented as a superposition of trigonometrical functions, $\cos(ET_\gamma/\hbar)$ with $T_\gamma \approx 2\pi/\omega_\perp$ and $2\pi/\omega_{\text{sh}}$, in a good approximation. We shall confirm in §6 that this supershell pattern in fact arises from an interference effect between two families of classical periodic orbits with periods $T_\gamma \approx 2\pi/\omega_{\text{sh}}$ and $2\pi/\omega_\perp$.

It should be recalled here that the important factor from the point of view of gaining the shell-structure energy is not the heights of the maxima but the depths of the minima in the oscillating level density. Needless to say, the minima in Fig. 3 correspond to the closed-shell configurations

with respect to the SD major shell quantum number N_{sh} . Let us notice how the depths of the minima change as functions of λ_{30} . Then we find that the minima associated with the odd- N_{sh} closures becomes shallower as λ_{30} increases, whereas those at the even- N_{sh} closures are tough. Consequently, the odd-even staggering of the minima with respect to the N_{sh} quantum number develops with increasing λ_{30} . Possible mechanisms of the enhancement of the supershell structure will be discussed in §6.

§5. CLASSICAL ANALYSIS

In this section, we discuss the classical-mechanical properties of the single-particle motion in the reflection-asymmetric SD potential defined in the preceding section.

5.1. Poincaré Map

Let us examine classical phase space structure by plotting the Poincaré map.^[21] Since our Hamiltonian is axially symmetric, it reduces to a two-dimensional one with the cylindrical coordinates (ρ, z) and with a definite angular momentum $p_\varphi = m$;

$$H = \frac{1}{2} (p_\rho^2 + p_z^2) + V_{\text{eff}}(\rho, z; m) , \quad (5.1)$$

where

$$V_{\text{eff}}(z, \rho; p_\varphi) = \frac{p_\varphi^2}{2\rho^2} + \frac{4\rho^2 + z^2}{2} - \lambda_{30} \sqrt{\frac{7}{4\pi}} \frac{z^3 - 6z\rho^2}{\sqrt{4\rho^2 + z^2}} . \quad (5.2)$$

We can examine the Poincaré map for each value of m . It is convenient to choose the Poincaré section Σ as the surface with $p_\rho = 0$, which is intersected by any trajectory. Figure 4 shows calculated Poincaré maps (z, p_z) for the Hamiltonian (4.1) with various values of the octupole deformation parameter λ_{30} . We see that the system is quasi-integrable for small λ_{30} and almost all the phase space is foliated by KAM tori. With increasing λ_{30} , however, chaotic regions begin to spread out from the hyperbolic points. Figure 5 shows Poincaré maps for different values of p_φ . We note that the phase space volume corresponding to the (ρ, z) degrees of freedom contract and the system becomes more regular as p_φ increases.

Figure 6 shows Poincaré maps for the surface of section (ρ, p_ρ) with $z = 0$. The origin corresponds to the linear orbit along the z -axis and the structure around it is exhibited.

5.2. Periodic Orbits and Their Bifurcations

While all trajectories are periodic when $\lambda_{30} = 0$, only very limited trajectories remain as periodic orbits when $\lambda_{30} \neq 0$. In the Poincaré section, centers of tori correspond to stable periodic orbits, while saddles to unstable ones. We calculate the periodic orbits by the monodromy method proposed by Baranger, Davies and Mahoney.^[27] In this method, periodic orbits are found in an iterative manner starting with approximate closed curves. By gradually changing λ_{30} , we can use the periodic orbits found for a slightly smaller value of λ_{30} as inputs for this procedure. Figure 7 shows short periodic orbits for Hamiltonian (4.1) with $\lambda_{30} = 0.4$ obtained in this way. Also shown in Fig. 8 are planar orbits for $\lambda_{30} = 0.3 \sim 0.4$. As λ_{30} increases, the phase space structure becomes more complicated due to bifurcations of stable periodic orbits. For example, a period-tripling bifurcation of orbit A occurs at $\lambda_{30} \simeq 0.36$. Thereafter, orbit A bifurcates into orbits 3A (triple traversal of orbit A), E and F. In the Poincaré map (see Fig. 4) for $\lambda_{30} = 0.37$, we can see three resonant island chains surrounding the central KAM torus, which are associated with the newly-born periodic orbits E and F. Likewise, a period-doubling bifurcation of orbit B occurs at $\lambda_{30} \simeq 0.4$, from where orbit B bifurcates into orbits 2B (double traversal of orbit B) and K. Many higher order bifurcations occur almost everywhere in regular regions of the phase space. Properties of the calculated periodic orbits are summarized in Table I for several values of λ_{30} .

§6. SEMICLASSICAL ANALYSIS

6.1. The Cause of the Enhancement of the Supershell Effect

As mentioned in §2.4, strength of shell effect associated with a periodic orbit is mainly determined by degeneracy and stability of the orbit. Let us discuss how these properties change when the octupole deformation is added to the SD oscillator. When $\lambda_{30} = 0$, orbits with the period $2\pi/\omega_{\text{sh}}$ and those with the period $2\pi/\omega_{\perp}$ have different degeneracies, 4 and 2, respectively. Therefore, the shell effect originating from the former families of orbits is much stronger than that from the latter. As a result, the interference effect between the two families of periodic orbit, *i.e.*, the supershell effect, is rather weak. When $\lambda_{30} \neq 0$, in general, degeneracies of the orbits reduce to 1. (For orbits D and A' in Fig. 7 having special symmetry, the degeneracy is 0.) Thus,

generally speaking, shell effects at $\lambda_{30} \neq 0$ are expected to become weaker in comparison with those in the $\lambda_{30} = 0$ limit, and it seems hard to understand the enhancement mechanism of the supershell effect at $\lambda_{30} \neq 0$ in terms of the degeneracy property.

Next, let us consider the other factor, *i.e.*, stability of periodic orbits. In Fig. 9 we show calculated values of $\text{Tr}M$ for relevant orbits as functions of λ_{30} . At $\lambda_{30} = 0$, orbits with the period $2\pi/\omega_{\text{sh}}$ are resonant ($\text{Tr}M = 2$), while orbits with the period $2\pi/\omega_{\perp}$ are non-resonant and take $\text{Tr}M = -2$. With increasing λ_{30} , $\text{Tr}M$ for orbits 2A(double traversal of orbit A) and B decrease and deviate from 2. At $\lambda_{30} \simeq 0.4$, a period-doubling bifurcation of orbit B occurs; a new stable orbit K is created and orbit B becomes unstable. Orbits C and C' are unstable for $\lambda_{30} > 0$ and their values of $\text{Tr}M$ become larger as λ_{30} increases. According to the argument in §2.4, we thus expect that the contributions of these orbits to the shell effect decrease with increasing λ_{30} . On the other hand, orbit A is stable and its $\text{Tr}M$ value approaches towards 2 as λ_{30} increases. This implies that the contribution of orbit A becomes more important. In this way, relative magnitude of the amplitude factors between the two families of orbits with the period $\approx 2\pi/\omega_{\text{sh}}$ and $\approx 2\pi/\omega_{\perp}$ changes so that the interference effect between them becomes stronger.

The above discussion is based on the expression (2.11) obtained by the SPA. We should note, however, that our classical phase space contains both regular and chaotic regions, *i.e.*, our system is a mixed system. As is well known, such a system is abundant in the resonance regions where the SPA breaks down, so that the amplitude factors $A_{n\gamma}$ should be evaluated by means of a more sophisticated method beyond the SPA, *e.g.*, the uniform approximation.^[28] This is an interesting future subject, and we expect that the above consideration will remain valid, as long as a qualitative feature is concerned, even when nonlinear effects beyond the SPA are taken into account.

6.2. Scaling Properties and Fourier Analysis

By virtue of the scaling property, Eq. (4.5), the following scaling rules hold;

$$\left\{ \begin{array}{l} S_{\gamma}(E) = ET_{\gamma} , \\ \bar{g}(E) = E^2 \bar{g}(1) , \\ A_{\gamma}(E) = E^{d_{\gamma}/2} A_{\gamma}(1) , \end{array} \right. \quad (6.1)$$

where d_γ denotes the degeneracy of orbit γ ; $d_\gamma = 1$ for a general orbit and 0 for an isolated orbit like D or A' in Fig. 7.

Let us consider the Fourier transform

$$P(s) = \int dE e^{isE} E^{-d/2} g(E) \quad (6.2)$$

of the level density $g(E)$ multiplied by $E^{-d/2}$. (The factor $E^{-d/2}$ is attached here to compensate for the energy dependence of the amplitude factor A_γ ; see below.) If one insert the exact level density $g(E) = \sum_n \delta(E - E_n)$, it becomes

$$P^{(\text{qm})}(s) = \sum_n E_n^{-d/2} e^{isE_n} \quad (6.3)$$

This quantity can be evaluated with the use of the eigenvalues obtained by a quantum mechanical calculation. On the other hand, if we insert the semiclassical level density (2.4) in (6.2) and put $d = 1$ appropriate to non-isolated orbits, then we obtain

$$P^{(\text{cl})}(s) = \bar{P}(s) + \sum_{n,\gamma} A_{n\gamma}(1) e^{i\pi\mu_{n\gamma}/2} \delta(s - nT_\gamma) . \quad (6.4)$$

Here $\bar{P}(s)$ comes from $\bar{g}(E)$ and has a peak at $s = 0$ associated with the orbits of zero length. On the other hand, the second term on the r.h.s. gives rise to sharp peaks at $s = nT_\gamma$ associated with the classical periodic orbits γ with periods T_γ (and their multiple traversals). Note that, owing to the scaling property (6.1), periods T_γ of the primitive orbits are equal to action $S_\gamma(1)$ calculated at $E = 1$. If the trace formula is valid, one expect $P^{(\text{cl})} \simeq P^{(\text{qm})}$. Thus, we can extract information about classical periodic orbits by calculating $P^{(\text{qm})}$. Namely, the amplitude factors and the Maslov phases of the periodic orbits may be obtained from absolute values and arguments of $P^{(\text{qm})}(s)$, respectively. [12], [31]

Now, let us evaluate the Fourier transform (6.3). Since the summation is taken over a finite number of quantum levels in practice, we introduce the Gaussian cut-off and define a smoothed version of it;

$$P_{\Delta s}(s) = \int ds' P(s') f((s' - s)/\Delta s) \quad (6.5)$$

where $f(x)$ is Gaussian $f(x) = \exp(-x^2/2)$. For (6.3) and (6.4), we obtain

$$P_{\Delta s}^{(\text{qm})}(s) = \sum_n E_n^{-d/2} e^{isE_n} f(E_n/E_{\text{max}}) , \quad (6.6)$$

$$P_{\Delta s}^{(\text{cl})}(s) = \bar{P}_{\Delta s}(s) + \sum_{n,\gamma} A_{n,\gamma}(1) e^{i\pi\mu_{n\gamma}} f((s - nT_\gamma)/\Delta s) , \quad (6.7)$$

where $E_{\max} = 1/\Delta s$.

We calculate the eigenvalues by a matrix diagonalization method with the deformed oscillator bases, and use the lower part of the resulting spectrum. Figure 10 shows the absolute value of $P_{\Delta s}^{(\text{qm})}(s)$ for $\lambda_{30} = 0.2 \sim 0.4$ calculated with $E_{\max} = 15\hbar\omega_{\text{sh}}(\lambda_{30})$. The loci of the periods of classical periodic orbits and their multiple traversals are indicated by arrows in the figures. We see nice correspondence between the peaks of $P(s)$ and the periods of classical periodic orbits. Almost all peaks can be explained in terms of the classical orbits, indicating that the properties of quantum spectrum is characterized mostly by classical periodic orbits.

Next, let us notice the λ_{30} dependence of $P^{(\text{qm})}(s)$. In Fig. 10 we see that the peak at $s \simeq 2\pi/\omega_{\text{sh}}$ decreases, while the peak at $s \simeq \pi/\omega_{\text{sh}}$ grows up with increasing λ_{30} . Since heights of the peaks in $P^{(\text{qm})}(s)$ indicate intensities coming from the corresponding periodic orbits, this implies that the contributions from the orbits with the period $\approx 2\pi/\omega_{\text{sh}}$ becomes increasingly important as λ_{30} increases. The change in relative intensity as a function of λ_{30} between the two families of periodic orbit seen in Fig. 10 may be responsible for the enhancement of the supershell effect in the reflection-asymmetric SD potential, in accordance with our discussion in the preceding subsection.

6.3. Quantum Signature of Bifurcations

In order to see how the bifurcations (resonances) of periodic orbits affect the magnitudes of the Fourier amplitudes, let us evaluate the heights of the peak as functions of λ_{30} at the periods of the classical orbits. As examples, we take the period-tripling and the period-5-upling bifurcations of orbit A, which occur at $\lambda_{30} = 0.36$ and 0.25 , respectively. In Fig. 11 are plotted the calculated values of $\bar{P}^{(\text{qm})}(s)$ as functions of λ_{30} at specific values of s that correspond to the periods of three- and five-fold traversals of orbit A. In accordance with the argument given below Eq. (2.10), we find that the heights of the peaks indeed exhibit supremes near the bifurcation (resonance) points, but little delays are observed for both cases. To account for this delay, it may be necessary to go beyond the SPA.

6.4. Angular Momentum Decomposition of the Trace Formula

As our system is axially symmetric, the angular momentum about the symmetry axis p_φ is a good quantum number. Thus, the level density can be decomposed as $g(E) = \sum_m g(E; m)$ with m denoting the angular momentum quantum number. Let us derive a semiclassical expression of $g(E; m)$. Writing the three-dimensional coordinate vector as $\mathbf{q} = (\mathbf{Q}, \varphi)$ with $\mathbf{Q} = (\rho, z)$, the Green function may be decomposed as

$$\begin{aligned} G(\mathbf{q}'', \mathbf{q}'; E) &= \sum_{n=-\infty}^{\infty} G((\mathbf{Q}'', \psi + 2n\pi), (\mathbf{Q}', 0); E) \\ &= \sum_n \int dM e^{i(\psi + 2n\pi)M} \tilde{G}(\mathbf{Q}'', \mathbf{Q}'; E, M) \\ &= \sum_{m=-\infty}^{\infty} e^{im\psi} \tilde{G}(\mathbf{Q}'', \mathbf{Q}'; E, m), \end{aligned} \quad (6.8)$$

where \tilde{G} denotes a Fourier transform of G with respect to $\psi = \varphi' - \varphi$, and where the Poisson sum formula is used in obtaining the last expression. Taking the trace of Eq. (6.8), one can derive the trace formula for $g(E; m)$ in a way similar to (2.4);^[32]

$$\begin{aligned} g(E; m) &= -2 \text{Im} \int d\mathbf{Q} \rho \tilde{G}(\mathbf{Q}, \mathbf{Q}; E, m) \\ &\simeq \bar{g}(E; m) + \frac{1}{\pi \hbar} \sum_{n, \alpha} \frac{\tau_\alpha}{\sqrt{|W_{n\alpha}|}} \cos(n\sigma_\alpha(E)/\hbar - (\pi/2)\mu_{n\alpha}), \end{aligned} \quad (6.9)$$

where σ_α denotes the action integrals along the two dimensional closed orbits α in the (ρ, z) plane, τ_α the periods, and μ_α the Maslov phases. Using the symplectic property of the monodromy matrix, $W_{n\alpha}$ can be written as

$$W_{n\alpha} = \det(\mathbf{1} - M_\alpha^n) = 2 - \text{Tr}(M_\alpha^n), \quad (6.10)$$

where M_α is a 2×2 monodromy matrix. Note that, for symmetric self-retracing orbits, M_α^n is different from \tilde{M}_α^n appearing in Eq. (2.11). It is easily seen that, due to the reflection symmetry with respect to the z -axis, these orbits in the (ρ, z) plane have periods half of those in the three dimensional space.

Now, for $m = 0$, a scaling property holds so that we can use the Fourier transformation technique. Since the degeneracy of the orbits is zero in the two-dimensional space, we put $d = 0$ in Eq. (6.6). From the above consideration, one expect that the Fourier transform will exhibit

peaks, in addition to those corresponding to the periods of closed orbits, also at half of the periods of the three-dimensional symmetric self-retracing orbits. The results of calculation is shown in Fig. 12. Again we find a clear correspondence between peaks of the Fourier transform and periods of classical orbits. As expected, peaks appear also at half integer times the period of orbit A.

§7. CONCLUDING REMARKS

We have found a clear correspondence between the shell structure, *i.e.*, the oscillatory structure in the smoothed level density, and the classical periodic orbits for single-particle motions in a reflection-asymmetric SD oscillator potential. We have then shown that the supershell effect, *i.e.*, an interference effect between two families of the periodic orbits having periods approximately $2\pi/\omega_{\perp}$ and $2\pi/\omega_{\text{sh}}$, develops when the reflection-asymmetric deformation increases. This supershell effect is in clear correspondence with the odd-even effect in N_{sh} pointed out in Refs. [8] and [10]. Possible origins of this enhancement phenomena have been pointed out in connection with stabilities of the classical periodic orbits. Quantum signature of the period-tripling bifurcation of the shortest-period orbit is also pointed out.

It should be emphasized that our model Hamiltonian system is a mixed system where chaos and tori are intermixed; accordingly, period-multipling bifurcations occur, as we have seen, rather frequently when the reflection-asymmetric deformation parameter is varied. As is well known, the SPA breaks down at the bifurcation points so that we cannot use the Gutzwiller trace formula for the aim of calculating the smoothed level density. Instead, by virtue of the scaling property of our model Hamiltonian, we have been able to use the Fourier transformation technique to find the quantum-classical correspondence. Properties of the Gutzwiller amplitudes have been used only as a guide to qualitative discussions. It is an interesting future subject to investigate the problem discussed in this paper by using a more sophisticated method, like the uniform approximation, which goes beyond the SPA.

Acknowledgements

We thank W. Nazarewicz, M. Toda, Y. R. Shimizu, M. Matsuo, H. Aiba, S. Mizutori and T. Nakatsukasa for useful discussions.

APPENDIX A: DERIVATION OF EQ. (2.11)

Semiclassical expression of the Green function in a f -dimensional system is

$$G(\mathbf{q}'', \mathbf{q}'; E) = \sum \frac{2\pi}{(2\pi i \hbar)^{(f+1)/2}} \sqrt{|D_S|} \exp\left(\frac{i}{\hbar} S(\mathbf{q}'', \mathbf{q}'; E) - i\kappa \frac{\pi}{2}\right), \quad (\text{A1})$$

where $S(\mathbf{q}'', \mathbf{q}'; E)$ is the classical action $\int \mathbf{p} \cdot d\mathbf{q}$ along a trajectory connecting \mathbf{q}' and \mathbf{q}'' with energy E . The determinant D_S in the amplitude factor is given by

$$D_S = \begin{vmatrix} \partial^2 S / \partial \mathbf{q}'' \partial \mathbf{q}' & \partial^2 S / \partial E \partial \mathbf{q}' \\ \partial^2 S / \partial \mathbf{q}'' \partial E & \partial^2 S / \partial E^2 \end{vmatrix} \quad (\text{A2})$$

Let us consider a three-dimensional system ($f=3$) with axial symmetry, and define an orthogonal coordinate $\mathbf{q} = (\xi, \eta, \zeta)$ for each periodic orbit. We take ξ along the direction of the trajectory and η perpendicular to both ξ and the azimuthal direction φ . Differentiating the Hamilton-Jacobi equations

$$H(\mathbf{p}'' = \partial S / \partial \mathbf{q}'', \mathbf{q}'') = E, \quad (\text{A3a})$$

$$H(\mathbf{p}' = -\partial S / \partial \mathbf{q}', \mathbf{q}') = E, \quad (\text{A3b})$$

with respect to E and using $\dot{\eta} = \dot{\zeta} = 0$, one obtains

$$1 = \sum_i \frac{\partial H}{\partial p_i''} \frac{\partial^2 S}{\partial q_i'' \partial E} = \dot{\xi}'' S_{\xi'' E}, \quad (\text{A4a})$$

$$1 = - \sum_i \frac{\partial H}{\partial p_i'} \frac{\partial^2 S}{\partial E \partial q_i'} = -\dot{\xi}' S_{E \xi'}, \quad (\text{A4b})$$

where S_{xy} denotes $(\partial^2 S / \partial x \partial y)$. If one differentiates (A3a) and (A3b) with ξ' and ξ'' , respectively, one obtains

$$S_{q_i'' \xi'} = S_{\xi'' q_i'} = 0 \quad (\text{A5})$$

Thus, the determinant (A2) is written as

$$D_S = \begin{vmatrix} S_{EE} & S_{E\xi'} & S_{E\eta'} & S_{E\zeta'} \\ S_{\xi'' E} & 0 & 0 & 0 \\ S_{\eta'' E} & 0 & S_{\eta'' \eta'} & S_{\eta'' \zeta'} \\ S_{\zeta'' E} & 0 & S_{\zeta'' \eta'} & S_{\zeta'' \zeta'} \end{vmatrix} = -\frac{1}{\dot{\xi}''} \frac{1}{\dot{\xi}'} \begin{vmatrix} S_{\eta'' \eta'} & S_{\eta'' \zeta'} \\ S_{\zeta'' \eta'} & S_{\zeta'' \zeta'} \end{vmatrix}. \quad (\text{A6})$$

If we use coordinates (ξ, η, φ) which are generally not orthogonal, then we obtain

$$D_S = \frac{1}{J''\xi''} \frac{1}{J'\xi'} \begin{vmatrix} S_{\eta''\eta'} & S_{\eta''\psi} \\ S_{\psi\eta'} & S_{\psi\psi} \end{vmatrix}, \quad (\text{A7})$$

where J is Jacobian of the coordinate transformation, J' and J'' denoting its value at \mathbf{q}' and \mathbf{q}'' , respectively, and $\psi = \varphi'' - \varphi'$. Let us evaluate the trace of the Green function in the stationary phase approximation. As usual, the action integral along a closed path may be expanded about a stationary point $\bar{\mathbf{q}}$ as

$$\begin{aligned} S(\mathbf{q}, \mathbf{q}; E) &= S(\bar{\mathbf{q}}, \bar{\mathbf{q}}; E) + (\mathbf{q} - \bar{\mathbf{q}})^T \left[\frac{\partial S}{\partial \mathbf{q}''} + \frac{\partial S}{\partial \mathbf{q}'} \right]_{\mathbf{q}''=\mathbf{q}'=\bar{\mathbf{q}}} \\ &+ \frac{1}{2}(\mathbf{q} - \bar{\mathbf{q}})^T \left[\frac{\partial^2 S}{\partial \mathbf{q}'' \partial \mathbf{q}''} + 2 \frac{\partial^2 S}{\partial \mathbf{q}'' \partial \mathbf{q}'} + \frac{\partial^2 S}{\partial \mathbf{q}' \partial \mathbf{q}'} \right]_{\mathbf{q}''=\mathbf{q}'=\bar{\mathbf{q}}} (\mathbf{q} - \bar{\mathbf{q}}) + \dots \end{aligned} \quad (\text{A8})$$

The stationary phase condition requires the second term in the r.h.s. to vanish. This is nothing but the condition for the trajectory to be periodic, *i.e.*, $\mathbf{p}'' = \mathbf{p}'$. Taking the axial symmetry into account, we can rewrite Eq. (A8) as

$$S(\mathbf{q}, \mathbf{q}; E) = \bar{S}(E) + \frac{1}{2}W(\xi)\eta^2 + \dots, \quad (\text{A9})$$

where

$$W(\xi) = \det(\mathbf{1} - M) S_{\eta''\eta'} = (2 - \text{Tr} M) S_{\eta''\eta'}. \quad (\text{A10})$$

M being the (2×2) monodromy matrix for the periodic orbit (see §2.2). Performing the Gauss-Fresnel integral with respect to η , we finally obtain the following result

$$\begin{aligned} g_{\text{osc}}(E) &= \frac{1}{2\pi^2 \hbar^2} \text{Im} \sum \int d\varphi d\xi d\eta J \sqrt{|D_S|} \exp \left[\frac{i}{\hbar} (\bar{S} + \frac{1}{2}W(\xi)\eta^2) - i\kappa \frac{\pi}{2} \right] \\ &= \frac{4\pi}{(2\pi\hbar)^{3/2}} \sum_{\gamma} \frac{B_{\gamma}}{\sqrt{|2 - \text{Tr} M_{\gamma}|}} \cos(S_{\gamma}/\hbar - \mu_{\gamma}\pi/2), \end{aligned} \quad (\text{A11})$$

where $\mu_{\gamma} = \kappa_{\gamma} - \text{sign}(W_{\gamma})/2$ and

$$\begin{aligned} B_{\gamma} &= \oint_{\gamma} \frac{d\xi}{|\dot{\xi}|} \left| \begin{pmatrix} S_{\eta''\eta'} & S_{\eta''\psi} \\ S_{\psi\eta'} & S_{\psi\psi} \end{pmatrix} / S_{\eta''\eta'} \right|^{1/2} \\ &= \int_{\gamma} dt \left| \frac{\partial p_{\varphi}}{\partial \varphi''} \right|^{1/2} = \int_{\gamma} dt \left| \frac{\partial \varphi(t + T_{\gamma})}{\partial p_{\varphi}(t)} \right|^{-1/2}. \end{aligned} \quad (\text{A12})$$

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TABLES

TABLE I. Properties of the periodic orbits: periods T (in unit of $1/\omega_{\text{sh}}(0)$) and traces of the reduced monodromy matrices $\text{Tr}M$, evaluated for $\lambda_{30} = 0, 0.2, 0.3$ and 0.4 . Here, ' $n\gamma$ ' denote the n -fold traversal of the primitive orbit γ . For the isolated orbit A' and D, the monodromy matrix M has two unit-eigenvalues and the remaining four eigenvalues appear in pairs $(e^{\alpha_a}, e^{-\alpha_a})$ and $(e^{\alpha_b}, e^{-\alpha_b})$. These pairs are identical ($\alpha_a = \alpha_b$) for orbit D, but they differ from each other for orbit A'. Traces of these pairs are given for orbit A'.

λ_{30}	0		0.2		0.3		0.4	
orbit	T/π	$\text{Tr}M$	T/π	$\text{Tr}M$	T/π	$\text{Tr}M$	T/π	$\text{Tr}M$
A	1	-2	1.018	-1.778	1.037	-1.362	1.062	-0.758
2A	2	2	2.036	1.161	2.075	-0.145	2.125	-1.426
3A	3	-2	3.054	-0.286	3.112	1.560	3.187	1.838
E	—	—	—	—	—	—	3.181	1.546
F	—	—	—	—	—	—	3.183	2.292
B	2	2	1.999	1.816	1.998	1.017	1.995	-2.054
2B	4	2	3.999	1.300	3.997	-0.966	3.989	2.221
K	—	—	—	—	—	—	3.989	1.572
C	2	2	2.001	2.030	2.004	2.383	2.009	4.277
D	2	2	2.072	1.845	2.177	3.498	2.367	-16.317
A'	1	$\begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$	1.024	$\begin{Bmatrix} -1.777 \\ 1.992 \end{Bmatrix}$	1.050	$\begin{Bmatrix} -1.264 \\ 1.960 \end{Bmatrix}$	1.086	$\begin{Bmatrix} -0.499 \\ 1.869 \end{Bmatrix}$
C'	2	2	2.003	2.046	2.008	2.333	2.016	3.484

FIGURES

FIG. 1. Results of semiclassical calculation of the oscillating part of the level density for the SD oscillator model. a) The contribution from orbits with $T = 2\pi/\omega_z$, b) the contribution from orbits with $T = 2\pi/\omega_\perp$, and c) the supershell structure caused by the interference between the above two families of orbits.

FIG. 2. Oscillating parts of the level density for the Hamiltonian (4.4) with $\lambda_{30} = 0.2, 0.3$ and 0.4 , calculated by means of the Strutinsky method with the smoothing width $\gamma = 0.5\hbar\omega_{\text{sh}}$. The arrows indicate the minima associated with the even- N_{sh} closure.

FIG. 3. The oscillating part of the level density at $\lambda_{30} = 0.4$ obtained by the Strutinsky method (broken line) is compared with a superposition (solid line) of $\cos(ET_\gamma/\hbar)$ with the period T_γ evaluated for the classical orbit A ($T \approx 2\pi/\omega_\perp$) and that for orbits B, C, C', and 2A ($T \approx 2\pi/\omega_{\text{sh}}$) (see §5 for the properties of these classical orbits). Amplitudes and phases of these cosine functions are determined so that the solid line best agrees with the broken line, except that the energy dependence of the amplitudes is assumed to fulfill the relation in Eq. (6.1) determined by the scaling property of the system under consideration (see §6).

FIG. 4. Poincaré maps in the section (z, p_z) for the Hamiltonian (4.1) with $p_\varphi = 0$ and with $\lambda_{30} = 0.2 \sim 0.4$, defined by $p_\rho = 0$ and $\dot{p}_\rho < 0$.

FIG. 5. Poincaré maps in the section (z, p_z) for the Hamiltonian (4.1) with $\lambda_{30} = 0.4$ and with $p_\varphi/E = 0.2$ and 0.4 , defined by $p_\rho = 0$ and $\dot{p}_\rho < 0$.

FIG. 6. Poincaré maps in the section (ρ, p_ρ) for the Hamiltonian (4.1) with $p_\varphi = 0$ and with $\lambda_{30} = 0.2$ and 0.4 , defined by $z = 0$ and $p_z > 0$.

FIG. 7. Short periodic orbits for the Hamiltonian (4.1) with $\lambda_{30} = 0.4$. Upper part: Planar orbits in the plane containing the symmetric axis z . Lower part: A circular orbit in the plane perpendicular to the symmetry axis (A') and a three-dimensional orbit (C'). Their projections on the (x, y) plane and on the (z, y) plane are shown.

FIG. 8. Short planar orbits for the Hamiltonian (4.1) with $p_\varphi = 0$ and $\lambda_{30} = 0.3 \sim 0.4$.

FIG. 9. Traces of the reduced monodromy matrices $\text{Tr}\tilde{M}$ for the non-isolated periodic orbits shown in Fig. 7 (see text for their definitions).

FIG. 10. Fourier transform $P_{\Delta s}^{(\text{qm})}(s)$ of the level density $g(E)$ defined by Eq. (6.6), for $\lambda_{30} = 0.2, 0.3$ and 0.4 . Gaussian cut off is done with $E_{\text{max}} \equiv \hbar/\Delta s = 15\hbar\omega_{\text{sh}}(\lambda_{30})$. Arrows indicate periods of the classical periodic orbits (see Fig. 7) and of their repetitions. This figure is basically the same as Fig. 15 in our previous report; ^[11] but, accuracy of the numerical calculation is significantly improved so that the peak at $s/\pi \approx 1$ is now clearly seen. This improvement greatly facilitates the discussion on the classical-quantum correspondence (see text).

FIG. 11. Peaks heights in the Fourier transform defined by Eq. (6.6) at the periods of classical periodic orbit A and of its multiple traversals ($m=1,3,5$), plotted as functions of λ_{30} . Gaussian cut off is done with $E_{\text{max}} = 12\hbar\omega_{\text{sh}}(\lambda_{30})$. Two arrows represent the period-tripling and period-5-upling bifurcation points.

FIG. 12. Fourier transforms of the level density $g(E, m)$ in the $m=0$ subspace for $\lambda_{30}=0.2, 0.3$ and 0.4 .

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